Summation Inequalities To Bounded Real Lemmas of Discrete-Time Systems With Time-Varying Delay

Chuan-Ke Zhang, *Member, IEEE,* Yong He, *Senior Member, IEEE,* Lin Jiang, *Member, IEEE,*
Min Wu, *Senior Member, IEEE*, and Hong-Bing Zeng

***Abstract*—Summation inequality is an important technique for analysis of discrete-time systems with a time-varying delay. It seems that from the literature a tighter inequality usually leads to a less conservative criterion. Based on H_∞ performance analysis problem, this note presents different findings on the relationship between the conservatism of bounded real lemma (BRL) and the tightness of summation inequality. Firstly, the BRL obtained by the Wirtinger-based inequality (WBI) is not always less conservative than the one by the Jensen-based inequality although the WBI is tighter. Secondly, the WBI is tighter than a general free-matrix-based inequality (GFMBI) developed in this note, while the BRL obtained via the GFMBI is less conservative than the WBI-based BRL. Finally, a numerical example is given to demonstrate those findings.**

***Index Terms*—Discrete-time system, time-varying delay, summation inequality, bounded real lemma**

I. INTRODUCTION

Since time-varying delays arising in discrete-time systems may cause undesired dynamics such as performance degradation and even instability, the investigation of their influences on the dynamic performances has become a hot topic in the field of control theory [1]–[27]. Among the methods developed for this topic, the Lyapunov functional method is the most popular one [9]. The investigation considering delay bound information, known as delay-dependent analysis, can lead to less conservative criteria compared with the delay-independent analysis. To obtain delay-dependent criteria, a double summation term defined in (2) is frequently applied during the construction of Lyapunov functional, and it introduces a single summation term, $\mathcal{S}(k)$ defined in (3), into its forward difference. In order to express the criteria in the form of tractable linear matrix inequalities (LMIs), a challenging problem arising is how to estimate this summation term [10].

So far, many methods have been proposed to estimate the single summation term, such as the free-weighting matrix (FWM) approach [5], the Jensen-based inequality (JBI) [7], the Wirtinger-based inequalities (WBIs) [8]–[10], the auxiliary-function-based inequality [11], and the FWM-based inequalities [12], [13]. The FWM approach and the JBI may lead to the equivalent criteria [1]. Compared with the FWM approach, the JBI leads to the criteria with less complexity such that it was widely used, especially after the development of reciprocally convex lemma that provides an effective way to handle the time-varying delay included in the denominators [28]. For continuous time-delay systems, the Wirtinger-based integral inequality was found to be less conservative than the Jensen integral inequality [30]. Based on this viewpoint, three types of WBIs, a Abel-lamma-based summation inequality (ABI), and an auxiliary-function-based inequality were reported to improve the JBI-based results. From the literature, the gap between two sides of inequality that indicates the tightness of inequality is an important factor related to the conservatism of criteria; and it seems to be predictable that a tighter inequality leads to a less conservative criterion [8]–[11].

From the procedure of criterion-development, it is found that the conservatism of criterion is dependent on not only the estimation of the $\mathcal{S}(k)$ but also the choice of Lyapunov functional [8]. Then, a question arises: *Does a tighter inequality always lead to a less conservative criterion when the same Lyapunov functional is used?* Answering this question is important to check the contribution of different techniques on conservatism-reduction. It motivates the current study.

In this note, based on H_∞ performance analysis of a discrete-time system with a time-varying delay, two findings different from the previous experience on the relationship between the conservatism of bounded real lemma (BRL) and the tightness of summation inequality are presented. First, the BRL obtained by the WBI is not always less conservative than the one by the JBI although the WBI is tighter than the JBI. Second, the WBI is tighter than the general free-matrix-based inequality (GFMBI) derived in this note, while the GFMBI can lead to less conservative BRLs. Finally, those interesting findings are verified via a numerical example.

Notations: Throughout this note, the superscripts T and -1 mean the transpose and the inverse of a matrix, respectively; $\|\cdot\|$ refers to the Euclidean vector norm; $P>0$ (≥ 0) means that P is a real symmetric and positive-definite (semi-positive-definite) matrix; and the symmetric term in a symmetric matrix is denoted by $*$. Matrices, if their dimensions are not explicitly stated, are assumed to be compatible for algebraic operations.

II. PROBLEM FORMULATION AND PRELIMINARIES

Consider the following linear discrete-time system with a time-varying delay:

$$\begin{cases} x(k+1) = Ax(k) + A_d x(k-d(k)) + B\omega(k), \ \forall k \geq 0 \\ x(k) = \phi(k), \qquad \forall k \in [-h, 0] \\ z(k) = Cx(k) \end{cases} \tag{1}$$

where $x(k)$, $z(k)$, $\omega(k)$, and $\phi(k)$ are the system state, the controlled output, the disturbance, and the initial condition, respectively; A, A_d, B, and C are the system matrices; $d(k) \in [0, h]$ is the time-varying delay, and h is a positive integer.

The following double summation term is frequently used in the Lyapunov functional to derive delay-dependent criteria:

$$V_r(k) = \sum_{i=-h}^{-1} \sum_{j=k+i}^{k-1} \eta^T(j) R \eta(j) \tag{2}$$

This work is supported partially by the National Natural Science Foundation of China under Grant Nos. 61503351, 51428702, 61573325, and 61210011, and by the Hubei Provincial Natural Science Foundation of China under Grant 2015CFA010. (Corresponding author: Yong He)

C.-K. Zhang is with the School of Automation, China University of Geosciences, Wuhan 430074, China, and also with the Department of Electrical Engineering & Electronics, University of Liverpool, Liverpool L69 3GJ, United Kingdom.

Y. He and M. Wu are with the School of Automation, China University of Geosciences, Wuhan 430074, China. (Yong He: heyong08@cug.edu.cn)

L. Jiang is with the Department of Electrical Engineering & Electronics, University of Liverpool, Liverpool L69 3GJ, United Kingdom.

H.-B. Zeng is with School of Electrical and Information Engineering, Hunan University of Technology, Zhuzhou 412007, China.

where $R > 0$, and $\eta(k) = x(k+1) - x(k)$. Then the following term is introduced into its forward difference:

$$\mathcal{S}(k) := \sum_{i=a}^{b-1} \eta^T(i) R \eta(i), \ \begin{cases} a = k - d(k), b = k & \text{or} \\ a = k - h, b = k - d(k) \end{cases} \quad (3)$$

Estimating $\mathcal{S}(k)$ via suitable inequalities is an important step to analyze system (1). It seems that a tighter inequality usually leads to a less conservative criterion from the literature [8]–[11]. Based on H_∞ performance analysis of system (1), this note further investigates the relationship between the tightness of summation inequalities and the conservatism of the BRLs, and some interesting findings are proposed.

Remark 1: Unlike the literature [23]–[27], which aims to derive a BRL with as small conservatism as possible, the main issue concerned in this note is to reveal new findings on the relationship between the tightness of inequalities and the conservatism of corresponding results. To solve this problem, not only the tightness comparison of inequalities but also the conservatism comparison of related criteria should be investigated theoretically. In this note, the BRLs are developed to carry out the conservatism comparison.

Definition 1: [23] For a given $\gamma > 0$, system (1) has H_∞ performance index γ if two conditions are satisfied: i) system (1) is asymptotical stability for $\omega(k) = 0$; and ii) the controlled output $z(k)$ satisfies $||z(k)|| < \gamma ||\omega(k)||$ for zero initial condition.

Lemma 1: For vectors β_1 and β_2, symmetric matrices R_1 and R_2, any matrix S satisfying $\begin{bmatrix} R_1 & S \\ * & R_2 \end{bmatrix} \geq 0$, and scalars $\alpha_1 \in (0, 1)$ and $\alpha_2 > 0$, the following inequalities hold

$$\tfrac{1}{\alpha_1} \beta_1^T R_1 \beta_1 + \tfrac{1}{1-\alpha_1} \beta_2^T R_2 \beta_2 \geq \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}^T \begin{bmatrix} R_1 & S \\ * & R_2 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} \quad (4)$$

$$2\beta_1^T S \beta_2 \leq \alpha_2 \beta_1^T R_1 \beta_1 + \alpha_2^{-1} \beta_2^T R_2 \beta_2 \quad (5)$$

Proof: Inequality (4) is recalled from [30] and derived based on the reciprocally convex lemma [28], and inequality (5) is obtained from (4) by setting $\alpha_2 = \frac{1-\alpha_1}{\alpha_1}$. ■

III. MAIN RESULTS

In this section, the tightness comparison of several summation inequalities is analyzed, and several BRLs are developed based on two types of Lyapunov functions. Then, new findings on the relationship between the tightness of inequalities and the conservatism of the BRLs are summarized after carrying out necessary theoretical analysis.

A. Summation inequalities and tightness comparison

Several existing summation inequalities, together with the GFMBI proposed in this note, are summarized as follows.

Lemma 2: For symmetric matrix $R > 0$ and Z_i, any matrices Z_3, L_i, and N_i, $i = 1, 2$, satisfying $\begin{bmatrix} Z_1 & Z_3 & N_1 \\ * & Z_2 & N_2 \\ * & * & R \end{bmatrix} \geq 0$, integers a and b satisfying $a < b$, and vectors g_1 and g_2, such that the summation terms concerned are well defined, the following inequalities hold

$$1) \quad \text{JBI [7]}: \ \mathcal{S}(k) \geq \frac{1}{l} \vartheta_1^T R \vartheta_1 \quad (6)$$

$$2) \quad \text{WBI [9]}: \ \mathcal{S}(k) \geq \frac{1}{l} \begin{bmatrix} \vartheta_1 \\ \vartheta_2 \end{bmatrix}^T \begin{bmatrix} R & 0 \\ 0 & 3\left(\frac{l+1}{l-1}\right) R \end{bmatrix} \begin{bmatrix} \vartheta_1 \\ \vartheta_2 \end{bmatrix} \quad (7)$$

$$\geq \frac{1}{l} \begin{bmatrix} \vartheta_1 \\ \vartheta_2 \end{bmatrix}^T \begin{bmatrix} R & 0 \\ 0 & 3R \end{bmatrix} \begin{bmatrix} \vartheta_1 \\ \vartheta_2 \end{bmatrix} \quad (8)$$

$$3) \quad \text{WBI [8]}: \ \mathcal{S}(k) \geq \frac{1}{l} \begin{bmatrix} \vartheta_1 \\ \vartheta_3 \end{bmatrix}^T \begin{bmatrix} R & 0 \\ 0 & 3R \end{bmatrix} \begin{bmatrix} \vartheta_1 \\ \vartheta_3 \end{bmatrix} \quad (9)$$

$$4) \quad \text{ABI [10]}: \ \mathcal{S}(k) \geq \frac{1}{l} \begin{bmatrix} \vartheta_1 \\ \vartheta_4 \end{bmatrix}^T \begin{bmatrix} R & 0 \\ 0 & 3\left(\frac{l-1}{l+1}\right) R \end{bmatrix} \begin{bmatrix} \vartheta_1 \\ \vartheta_4 \end{bmatrix} \quad (10)$$

$$\begin{aligned} 5) \ \text{FMBI [12]}: \ \mathcal{S}(k) &\geq 2 \begin{bmatrix} \vartheta_5 \\ \vartheta_5 \end{bmatrix}^T \begin{bmatrix} -N_1 & 0 \\ 0 & N_2 \end{bmatrix} \begin{bmatrix} \vartheta_1 \\ \vartheta_2 \end{bmatrix} \\ &\quad - l \vartheta_5^T \left(\frac{3Z_1 + Z_2}{3} \right) \vartheta_5 \end{aligned} \quad (11)$$

$$\begin{aligned} 6) \ \text{GFMBI}: \ \mathcal{S}(k) &\geq 2 \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}^T \begin{bmatrix} L_1 & 0 \\ 0 & L_2 \end{bmatrix} \begin{bmatrix} \vartheta_1 \\ \vartheta_2 \end{bmatrix} \\ &\quad - l \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}^T \begin{bmatrix} \Omega_1 & 0 \\ 0 & \bar{\Omega}_3 \end{bmatrix} \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} \quad (12) \\ &\geq 2 \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}^T \begin{bmatrix} L_1 & 0 \\ 0 & L_2 \end{bmatrix} \begin{bmatrix} \vartheta_1 \\ \vartheta_2 \end{bmatrix} \\ &\quad - l \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}^T \begin{bmatrix} \Omega_1 & 0 \\ 0 & \Omega_3 \end{bmatrix} \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} \quad (13) \end{aligned}$$

where

$$\vartheta_1 = x(b) - x(a), \quad \vartheta_2 = x(b) + x(a) - \sum_{i=a}^{b} \frac{2x(i)}{l+1}$$

$$\vartheta_3 = \frac{l-1}{l} x(b) + \frac{l+1}{l} x(a) - \sum_{i=a}^{b-1} \frac{2x(i)}{l}, \quad l = b - a$$

$$\vartheta_4 = x(b) + x(a) - \sum_{i=a+1}^{b-1} \frac{2x(i)}{l-1}, \ \vartheta_5 = \left[x^T(b), x^T(a), \sum_{i=a}^{b} \frac{x^T(i)}{l+1} \right]^T$$

$$\Omega_i = L_i R^{-1} L_i^T, i = 1, 2; \ \bar{\Omega}_3 = \frac{l-1}{3(l+1)} \Omega_2, \quad \Omega_3 = \frac{1}{3} \Omega_2$$

Proof of 6): Inspired by our previous work [13] and [34], GFMBIs (12) and (13) can be obtained based on the following relationships:

$$\begin{aligned} 0 &\leq \sum_{i=a}^{b-1} \begin{bmatrix} g_1 \\ f(i) g_2 \\ \eta(i) \end{bmatrix}^T \begin{bmatrix} \Omega_1 & L_1 R^{-1} L_2^T & -L_1 \\ * & \Omega_2 & -L_2 \\ * & * & R \end{bmatrix} \begin{bmatrix} g_1 \\ f(i) g_2 \\ \eta(i) \end{bmatrix} \quad (14) \\ &= \mathcal{S}(k) + l \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}^T \begin{bmatrix} \Omega_1 & 0 \\ 0 & \bar{\Omega}_3 \end{bmatrix} \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} - 2 \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}^T \begin{bmatrix} L_1 & 0 \\ 0 & L_2 \end{bmatrix} \begin{bmatrix} \vartheta_1 \\ \vartheta_2 \end{bmatrix} \\ &\leq \mathcal{S}(k) + l \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}^T \begin{bmatrix} \Omega_1 & 0 \\ 0 & \Omega_3 \end{bmatrix} \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} - 2 \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}^T \begin{bmatrix} L_1 & 0 \\ 0 & L_2 \end{bmatrix} \begin{bmatrix} \vartheta_1 \\ \vartheta_2 \end{bmatrix} \end{aligned}$$

where $f(i) = (2i - a - b + 1)/(b - a + 1)$. ■

Proposition 1: The relationships among summation inequalities (6)-(13) are summarized as follows.

1) Both WBIs (7)-(9) and ABI (10) are tighter than JBI (6) since the former includes extra positive terms [8]–[10];
2) WBI (7) is tighter than WBI (9) [13]; WBI (9) is tighter than WBI (8) [13]; and FMBI (11) includes WBI (8) [12];
3) WBI (7) is equivalent to ABI (10);
4) GFMBI (13) is tighter than FMBI (11);
5) WBI (7) is tighter than GFMBI (12);
6) WBI (8) is tighter than GFMBI (13).

Proof: 1) and 2) have been analyzed in the literature; 3) can be directly obtained considering the fact of $\vartheta_4 = \frac{l+1}{l-1} \vartheta_2$.

4) It follows $\begin{bmatrix} Z_1 & Z_3 & N_1 \\ * & Z_2 & N_2 \\ * & * & R \end{bmatrix} \geq 0$ that $N_i R^{-1} N_i^T \leq Z_i, i = 1, 2$. Setting $g_i = \vartheta_5, i = 1, 2$, $L_1 = -N_1$, and $L_2 = N_2$ yields

$$\begin{aligned} \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}^T \begin{bmatrix} \Omega_1 & 0 \\ 0 & \Omega_3 \end{bmatrix} \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} &= \vartheta_5^T \left(\frac{3N_1 R^{-1} N_1^T + N_2 R^{-1} N_2^T}{3} \right) \vartheta_5 \\ &\leq \vartheta_5^T \left(\frac{3Z_1 + Z_2}{3} \right) \vartheta_5 \end{aligned} \tag{15}$$

Thus,

$$\begin{aligned} &2 \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}^T \begin{bmatrix} L_1 & 0 \\ 0 & L_2 \end{bmatrix} \begin{bmatrix} \vartheta_1 \\ \vartheta_2 \end{bmatrix} - l \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}^T \begin{bmatrix} \Omega_1 & 0 \\ 0 & \Omega_3 \end{bmatrix} \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} \\ \geq &2 \begin{bmatrix} \vartheta_5 \\ \vartheta_5 \end{bmatrix}^T \begin{bmatrix} -N_1 & 0 \\ 0 & N_2 \end{bmatrix} \begin{bmatrix} \vartheta_1 \\ \vartheta_2 \end{bmatrix} - l \vartheta_5^T \left(\frac{3Z_1 + Z_2}{3} \right) \vartheta_5 \end{aligned}$$

Similar to the discussion in [34], the above shows that the gap between two sides of GFMBI (13) is smaller than that of FMBI (11). That is, GFMBI (13) is tighter than FMBI (11).

5) If setting $\beta_1 = g_1$, $S = L_1$, $\beta_2 = \upsilon_1$, $\alpha_2 = l$, $R_1 = \Omega_1$, and $R_2 = R$, then $\begin{bmatrix} R_1 & S \\ * & R_2 \end{bmatrix} \geq 0$ holds based on $R > 0$ and Schur complement, thus it follows (5) that

$$2 g_1^T L_1 \vartheta_1 \leq l g_1^T \Omega_1 g_1 + \frac{1}{l} \vartheta_1^T R \vartheta_1 \tag{16}$$

Similarly, if setting $\beta_1 = g_2$, $S = L_2$, $\beta_2 = \upsilon_2$, $\alpha_2 = l$, $R_1 = \bar{\Omega}_3$, and $R_2 = \frac{3(l+1)}{l-1} R$, then the following holds

$$2 g_2^T L_2 \vartheta_2 \leq l g_2^T \bar{\Omega}_3 g_2 + \frac{1}{l} \vartheta_2^T \frac{3(l+1)R}{l-1} \vartheta_2 \tag{17}$$

Thus, the following holds

$$\begin{aligned} &2 \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}^T \begin{bmatrix} L_1 & 0 \\ 0 & L_2 \end{bmatrix} \begin{bmatrix} \vartheta_1 \\ \vartheta_2 \end{bmatrix} - l \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}^T \begin{bmatrix} \Omega_1 & 0 \\ 0 & \bar{\Omega}_3 \end{bmatrix} \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} \\ \leq &\frac{1}{l} \begin{bmatrix} \vartheta_1 \\ \vartheta_2 \end{bmatrix}^T \begin{bmatrix} R & 0 \\ 0 & 3\frac{l+1}{l-1} R \end{bmatrix} \begin{bmatrix} \vartheta_1 \\ \vartheta_2 \end{bmatrix} \end{aligned} \tag{18}$$

The above shows that the gap of WBI (7) is smaller than that of GFMBI (12). That is, WBI (7) is tighter than GFMBI (12).

6) can be obtained via the similar proof procedure of 5). ■

Remark 2: As reported in [9], the fraction, $\frac{l+1}{l-1}$, in WBI (7) brings difficulty during the study of system with a time-varying delay, thus WBI (8) excluded this fraction via $\frac{l+1}{l-1} > 1$ is applied. Similarly, GFMBI (13) derived by excluding $-\frac{l-1}{l+1}$ in GFMBI (12) via $-\frac{l-1}{l+1} > -1$ can be applied for the time-varying delay case. Since the fraction, $\frac{l-1}{l+1}$, included in ABI (10) cannot be removed following the similar treatment, ABI (10) cannot be used directly for the time-varying delay case.

B. BRLs via the existing and the proposed inequalities

At first, based on two Lyapunov functionals, four BRLs of system (1) are obtained via the existing inequalities, such as JBI (6), WBI (8), and FMBI (11).

Theorem 1: For given h and γ, system (1) has H_∞ performance index γ if one of the following conditions holds

C1: (SLF+JBI) there exist symmetric matrices $\bar{P} > 0$, $Q \geq 0$, $R \geq 0$, and matrix S, such that the following holds:

$$\Phi_1 \geq 0, \qquad \Phi_2 \leq 0 \tag{19}$$

C2: (SLF+WBI) there exist symmetric matrices $\bar{P} > 0$, $Q \geq 0$, $R \geq 0$, and matrix T, such that the following holds:

$$\Phi_3 \geq 0, \qquad \Phi_4 \leq 0 \tag{20}$$

C3: (ALF+WBI) there exist symmetric matrices $P > 0$, $Q \geq 0$, $R \geq 0$, and matrix T, such that the following holds:

$$\Phi_3 \geq 0, \qquad \Phi_{5,j} \leq 0,\ j = 1, 2 \tag{21}$$

C4: (ALF+FMBI) there exist symmetric matrices $P > 0$, $Q \geq 0$, $R \geq 0$, $\bar{Z}_i = \begin{bmatrix} Z_{i1} & Z_{i3} \\ * & Z_{i2} \end{bmatrix}$ and matrices $L_i = \begin{bmatrix} L_{i1} \\ L_{i2} \end{bmatrix}$, such that the following holds:

$$\Phi_{6,i} \geq 0, i = 1, 2 \quad , \Phi_{7,j} \leq 0, j = 1, 2 \tag{22}$$

where the related notations are defined in Box I, and $\Phi_{k,1} = \Phi_k|_{d(k)=h}$, $\Phi_{k,2} = \Phi_k|_{d(k)=0}$, $k = 5, 7$.

Proof: Construct two types of Lyapunov functionals:

$$\text{SLF}: V_s(k) = x^T(k) \bar{P} x(k) + \sum_{i=k-h}^{k-1} x^T(i) Q x(i) + h V_r(k) \tag{29}$$

$$\text{ALF}: V_a(k) = \xi^T(k) P \xi(k) + \sum_{i=k-h}^{k-1} x^T(i) Q x(i) + h V_r(k) \tag{30}$$

where $\xi(k) = [x^T(k), \sum_{i=k-h}^{k-1} x^T(i)]^T$, $V_r(k)$ is defined in (2); and matrices $\bar{P} > 0$, $P > 0$, $Q \geq 0$, and $R \geq 0$.

Calculating the forward difference $\Delta V_s(k) = V_s(k+1) - V_s(k)$, applying JBI (6) to estimate $\mathcal{S}(k)$ appearing in $\Delta V_s(k)$, and using (4), together with $\Phi_1 \geq 0$, to handle $d(k)$ yield

$$\Delta V_s(k) + z^T(k) z(k) - \gamma^2 \omega^T(k) \omega(k) \leq \zeta_1^T(k) \Phi_2 \zeta_1(k)$$

where $\zeta_1(k) = [x^T(k), x^T(k - d(k)), x^T(k - h), \omega^T(k)]^T$ and Φ_2 is defined in (23). By following the similar lines as in [23], system (1) has H_∞ performance index γ if $\Phi_2 \leq 0$ holds. Then, Theorem 1.C1 is proved.

Calculating the forward difference of $V_s(k)$, applying WBI (8) to estimate $\mathcal{S}(k)$ appearing in $\Delta V_s(k)$, and using (4), together with $\Phi_3 \geq 0$, to handle $d(k)$ yield

$$\Delta V_s(k) + z^T(k) z(k) - \gamma^2 \omega^T(k) \omega(k) \leq \zeta_2^T(k) \Phi_4 \zeta_2(k)$$

where $\zeta_2(k) = [\zeta_1^T(k), \sum_{i=k-d(k)}^{k} \frac{x^T(i)}{l_1}, \sum_{i=k-h}^{k-d(k)} \frac{x^T(i)}{l_2}]^T$ with l_i defined in (27), and Φ_4 is defined in (24). System (1) has H_∞ performance index γ if $\Phi_4 \leq 0$ holds. Then, Theorem 1.C2 is proved.

Calculating the forward difference $\Delta V_a(k) = V_a(k+1) - V_a(k)$, applying WBI (8) to estimate $\mathcal{S}(k)$ appearing in $\Delta V_a(k)$, and using (4), together with $\Phi_3 \geq 0$, to handle $d(k)$ appearing in the denominators yield

$$\Delta V_a(k) + z^T(k) z(k) - \gamma^2 \omega^T(k) \omega(k) \leq \zeta_2^T(k) \Phi_5 \zeta_2(k)$$

where Φ_5 is defined in (24), and it can be rewritten as $\Phi_5 = \Gamma_1 + d(k) \Gamma_2$ with $\Gamma_i, i = 1, 2$ being time-independent matrix combinations. Thus, $\Phi_{5,j} \leq 0, j = 1, 2$ implies $\Phi_5 \leq 0$ [29], which further leads that system (1) has H_∞ performance index γ. Then Theorem 1.C3 is proved.

Calculating the forward difference of $V_a(k)$, setting suitable matrices $\bar{Z}_i = \begin{bmatrix} Z_{i1} & Z_{i3} \\ * & Z_{i2} \end{bmatrix}$ and $L_i = \begin{bmatrix} L_{i1} \\ L_{i2} \end{bmatrix}$ satisfying $\begin{bmatrix} \bar{Z}_i & L_i \\ * & R \end{bmatrix} \leq 0$, and applying FMBI (11) to estimate $\mathcal{S}(k)$ appearing yield

$$\Delta V_a(k) + z^T(k) z(k) - \gamma^2 \omega^T(k) \omega(k) \leq \zeta_2^T(k) \Phi_7 \zeta_2(k)$$

where Φ_7 is defined in (24), and can be rewritten as $\Phi_7 = \Gamma_3 + d(k) \Gamma_4$ with $\Gamma_i, i = 3, 4$ being time-independent matrix

Box I: Notations used in Theorems 1 and 2

$$\Phi_1 = \begin{bmatrix} R & S \\ * & R \end{bmatrix};\ \Phi_2 = (\bar{e}_s+\bar{e}_1)^T \bar{P}(\bar{e}_s+\bar{e}_1) - \bar{e}_1^T(\bar{P}-Q-C^TC)\bar{e}_1 - \bar{e}_3^T Q\bar{e}_3 + h^2\bar{e}_s^T R\bar{e}_s - \gamma^2\bar{e}_4^T\bar{e}_4 - \begin{bmatrix} \bar{e}_1-\bar{e}_2 \\ \bar{e}_2-\bar{e}_3 \end{bmatrix}^T \Phi_1 \begin{bmatrix} \bar{e}_1-\bar{e}_2 \\ \bar{e}_2-\bar{e}_3 \end{bmatrix};\ \Phi_3 = \begin{bmatrix} \tilde{R} & T \\ * & \tilde{R} \end{bmatrix} \tag{23}$$

$$\Phi_4 = \mathfrak{E}_1^T\bar{P}\mathfrak{E}_1 - e_1^T\bar{P}e_1 + \Xi_1 - \begin{bmatrix} \mathfrak{E}_2 \\ \mathfrak{E}_3 \end{bmatrix}^T \Phi_3 \begin{bmatrix} \mathfrak{E}_2 \\ \mathfrak{E}_3 \end{bmatrix};\ \Phi_5 = \Xi_2 - \begin{bmatrix} \mathfrak{E}_2 \\ \mathfrak{E}_3 \end{bmatrix}^T \Phi_3 \begin{bmatrix} \mathfrak{E}_2 \\ \mathfrak{E}_3 \end{bmatrix};\ \Phi_{6,i} = \begin{bmatrix} \bar{Z}_i & L_i \\ * & R \end{bmatrix};\ \Phi_7 = \Xi_2+\Xi_3+\Xi_3^T+\Xi_4;\ \Phi_9 = \Xi_2-\Xi_5-\Xi_5^T \tag{24}$$

$$\Xi_1 = e_1^T(Q+C^TC)e_1 - e_3^TQe_3 + h^2e_s^TRe_s - \gamma^2e_4^Te_4;\ \ \Xi_2 = \begin{bmatrix} \mathfrak{E}_1 \\ \mathfrak{E}_4 \end{bmatrix}^T P \begin{bmatrix} \mathfrak{E}_1 \\ \mathfrak{E}_4 \end{bmatrix} - \begin{bmatrix} e_1 \\ \mathfrak{E}_5 \end{bmatrix}^T P \begin{bmatrix} e_1 \\ \mathfrak{E}_5 \end{bmatrix} + \Xi_1;\ \ \Xi_3 = h\sum_{j=1}^{2}\mathfrak{E}_{5+j}^T[L_{j1},\ -L_{j2}]\mathfrak{E}_{1+j} \tag{25}$$

$$\Xi_4 = hd(k)\mathfrak{E}_6^T\left(\frac{3Z_{11}+Z_{12}}{3}\right)\mathfrak{E}_6 + h(h-d(k))\mathfrak{E}_7^T\left(\frac{3Z_{21}+Z_{22}}{3}\right)\mathfrak{E}_7;\ \ \Xi_5 = \sum_{j=1}^{2}\begin{bmatrix} e_{gj1} \\ e_{gj2} \end{bmatrix}^T \begin{bmatrix} L_{j1} & 0 \\ 0 & L_{j2} \end{bmatrix}\mathfrak{E}_{1+j};\ \tilde{R} = \begin{bmatrix} R & 0 \\ 0 & 3R \end{bmatrix} \tag{26}$$

$$\mathfrak{E}_1 = e_s+e_1;\ \mathfrak{E}_2 = \begin{bmatrix} e_1-e_2 \\ e_1+e_2-2e_5 \end{bmatrix};\ \mathfrak{E}_3 = \begin{bmatrix} e_2-e_3 \\ e_2+e_3-2e_6 \end{bmatrix};\ \begin{cases} \mathfrak{E}_4 = l_1e_5+l_2e_6-e_2-e_3 \\ \mathfrak{E}_5 = l_1e_5+l_2e_6-e_1-e_2 \end{cases};\ \begin{cases} \mathfrak{E}_6 = [e_1^T, e_2^T, e_5^T]^T \\ \mathfrak{E}_7 = [e_2^T, e_3^T, e_6^T]^T \end{cases};\ \begin{cases} l_1 = d(k)+1 \\ l_2 = h-d(k)+1 \end{cases} \tag{27}$$

$$\bar{e}_s = [A-I, A_d, 0, B], \bar{e}_i = [0_{n\times(i-1)n}, I, 0_{n\times(4-i)n}], i=1,2,3,4;\quad e_s = [A-I, A_d, 0, B, 0, 0], e_i = [0_{n\times(i-1)n}, I, 0_{n\times(6-i)n}], i=1,2,\cdots,6 \tag{28}$$

combinations. Thus, $\Phi_{7,j} \leq 0, j=1,2$ implies $\Phi_7 \leq 0$ [29], which further leads that system (1) has H_∞ performance index γ. Then Theorem 1.C4 is proved. ∎

In the following part, the proposed GFMBI (13) is used to develop BRLs based on the same Lyapunov functional (30). It is predictable that different GFMBI-based BRLs can be obtained by choosing different $g_i, i=1,2$ in (13) [chose to be a linear combination of vectors in $\zeta_2(k)$]. Two of them are given as follows.

Theorem 2: For given h and γ, system (1) has H_∞ performance index γ if one of the following conditions holds

C1: (ALF+GFMBI1) there exist symmetric matrices $P \geq 0$, $Q \geq 0$, $R \geq 0$, and matrix T, such that the following holds:

$$\Phi_{8,j} = \begin{bmatrix} \Phi_{5,j} - \mathfrak{E}_{4-j}^T\tilde{R}\mathfrak{E}_{4-j} & \mathfrak{E}_{4-j}^TT_j \\ * & -\tilde{R} \end{bmatrix} \leq 0, j=1,2 \tag{31}$$

C2: (ALF+GFMBI2) there exist symmetric matrices $P \geq 0$, $Q \geq 0$, $R \geq 0$, and matrices L_{j1} and $L_{j2}, j=1,2$, such that the following holds:

$$\begin{bmatrix} \bar{\Phi}_{9,j} & he_v^TL_{j1} & he_v^TL_{j2} \\ * & -R & 0 \\ * & * & -3R \end{bmatrix} \leq 0, j=1,2 \tag{32}$$

where $T_1 = T^T, T_2 = T$, $e_v = [\mathfrak{E}_2^T, \mathfrak{E}_3^T, e_6^T]^T$, $\bar{\Phi}_{9,j} = \Phi_{9,j}|_{e_v=e_{gj1}=e_{gj2}}$, $\Phi_{9,1} = \Phi_9|_{d(k)=h}$, $\Phi_{9,2} = \Phi_9|_{d(k)=0}$, and the other notations are defined in Box I.

Proof: Let e_{gij} be suitable linear combinations of e_i defined in (28), $g_i = e_{g1i}\zeta_2(k), L_i = L_{1i}, i=1,2$ for estimating $\mathcal{S}(k)$ with $a = k-d(k), b=k$, and $g_j = e_{g2j}\zeta_2(k), L_j = L_{2j}, j=1,2$ for estimating $\mathcal{S}(k)$ with $a=k-h, b=k-d(k)$. Then calculating the forward difference of $V_a(k)$, and applying GFMBI (13) to estimate $\mathcal{S}(k)$ appearing yield

$$\Delta V_a(k) + z^T(k)z(k) - \gamma^2\omega^T(k)\omega(k) \leq \zeta_2^T(k)\Psi\zeta_2(k)$$

where

$$\begin{aligned} \Psi =& \\ &\Phi_9 + hd(k)\left[e_{g11}^TL_{11}R^{-1}L_{11}^Te_{g11} + e_{g12}^TL_{12}(3R)^{-1}L_{12}^Te_{g12}\right] \\ &+h[h-d(k)]\left[e_{g21}^TL_{21}R^{-1}L_{21}^Te_{g21} + e_{g22}^TL_{22}(3R)^{-1}L_{22}^Te_{g22}\right] \end{aligned} \tag{33}$$

It can be rewritten as $\Psi = \Gamma_5 + d(k)\Gamma_6$ with $\Gamma_i, i=5,6$ being time-independent matrix combinations. Thus, $\Psi \leq 0$ holds if $\Psi|_{d(k)=h} \leq 0$ and $\Psi|_{d(k)=0} \leq 0$, i.e.,

$$\Phi_{9,j} + h^2\left[e_{gj1}^TL_{j1},\ e_{gj2}^TL_{j2}\right]\tilde{R}^{-1}\left[e_{gj1}^TL_{j1},\ e_{gj2}^TL_{j2}\right]^T \leq 0 \tag{34}$$

Therefore, (34) implies $\Psi \leq 0$, which further leads that system (1) has H_∞ performance index γ.

On the one hand, let $e_{gj1} = e_{gj2} = \begin{bmatrix} \mathfrak{E}_2 \\ \mathfrak{E}_3 \end{bmatrix}$, $T = \begin{bmatrix} T_1 & T_3 \\ T_2 & T_4 \end{bmatrix}$, and

$$L_{11} = \tfrac{1}{h}\left[R, 0, T_1, T_3\right]^T,\quad L_{12} = \tfrac{1}{h}\left[0, 3R, T_2, T_4\right]^T \tag{35}$$

$$L_{21} = \tfrac{1}{h}\left[T_1^T, T_2^T, R, 0\right]^T,\quad L_{22} = \tfrac{1}{h}\left[T_3^T, T_4^T, 0, 3R\right]^T \tag{36}$$

After simple algebraic calculation, (34) can be rewritten as

$$\Phi_{5,j} - \mathfrak{E}_{4-j}^T\tilde{R}\mathfrak{E}_{4-j} + \mathfrak{E}_{4-j}^TT_j\tilde{R}^{-1}T_j^T\mathfrak{E}_{4-j} \leq 0 \tag{37}$$

which is equivanlent to $\Phi_{8,j} \leq 0$ based on Schur complement. Thus, Theorem 2.C1 can be proved.

On the other hand, if setting $e_{gj1} = e_{gj2} = e_v$, then (32) implies (34) based on Schur complement. Thus, Theorem 2.C2 can be proved. ∎

Remark 3: Compared with Theorem 1.C3, Theorem 2.C2 has less conservatism (the proof will be given in the next subsection), while it keeps the same number of decision variables, $7n^2+2n$. That is, by choosing suitable $e_{gij}, L_{ij}, i,j=1,2$, the conservatism-reduction can be achieved through the proposed GFMBI without requiring additional decision variables. It is different from most existing inequalities, which usually achieve the reduction of conservatism at the cost of increase of variables [11]–[13], [31], [32].

Remark 4: Compared with four conditions of Theorem 1 and Theorem 2.C1, Theorem 2.C2 may lead to less conservative results. During the proof of Theorem 2.C2, $e_{gij}, i,j=1,2$ are chose to include an extra matrix e_6, which means that, after estimating $\mathcal{S}(k)$ by using GFMBI (13), several new cross terms related to $\omega(k)$ are introduced into the LMI conditions. To the best of authors' knowledge, no $\omega(k)$-dependent cross term will be introduced by using all existing inequalities, such as (6)-(11). As reported in [33], those cross terms may be helpful to relax the constraint of LMI-based conditions so as to reduce the conservatism.

C. *New findings from conservatism analysis*

This part will reveal two new findings on the relationship between the conservatism of BRL and the tightness of summation inequality, different from the previous experience from literature.

The first one is that *a tighter inequality does not always lead to a less conservative criterion, even under the same Lyapunov function.* This finding is concluded from the conservatism comparison of Theorem 1.C1 and Theorem 1.C2, which are all obtained by $V_s(k)$. Although the WBI is tighter than JBI as mentioned in Proposition 1, it can be proved that Theorem 1.C1 and Theorem 1.C2 are equivalent, represented as follows.

Proposition 2: For given h and γ, there exist feasible solutions of LMI (19) if and only if there exist feasible solutions of LMI (20).

Proof: Firstly, setting $T = \begin{bmatrix} S & S_2 \\ S_3 & S_4 \end{bmatrix}$ and carrying out simple calculations yield $\Phi_4 = \mathfrak{E}_e^T \bar{\Phi}_4 \mathfrak{E}_e$ with

$$\bar{\Phi}_4 = \begin{bmatrix} \Phi_2 & \Pi_1 \\ * & \Pi_2 \end{bmatrix}, \ \Pi_1 = -\mathfrak{E}_1 \begin{bmatrix} 0 & S_2 \\ S_3^T & 0 \end{bmatrix}, \ \Pi_2 = -\begin{bmatrix} 3R & S_4 \\ * & 3R \end{bmatrix}$$
$$\mathfrak{E}_e = [e_1^T, e_2^T, e_3^T, (e_1+e_2-2e_5)^T, (e_2+e_3-2e_6)^T]^T$$

Therefore

$$\Phi_4 \leq 0 \Longleftrightarrow \bar{\Phi}_4 \leq 0 \tag{38}$$

Secondly, if the matrices $(\bar{P}, Q, R, S)$ are feasible solutions of (19), then

$$\begin{cases} \Phi_1 \geq 0 \Longrightarrow \begin{bmatrix} \tilde{R} & \begin{bmatrix} S & 0 \\ 0 & 0 \end{bmatrix} \\ * & \tilde{R} \end{bmatrix} \geq 0 \Longrightarrow \Phi_3|_{T=\left[\begin{smallmatrix} S & 0 \\ 0 & 0 \end{smallmatrix}\right]} \geq 0 \\ \Phi_2 \leq 0 \Longrightarrow \begin{bmatrix} \Phi_2 & \mathfrak{E}_1 \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\ * & -3\begin{bmatrix} R & 0 \\ 0 & R \end{bmatrix} \end{bmatrix} \leq 0 \Longrightarrow \bar{\Phi}_4|_{T=\left[\begin{smallmatrix} S & 0 \\ 0 & 0 \end{smallmatrix}\right]} \leq 0 \\ \qquad \Longrightarrow \Phi_4|_{T=\left[\begin{smallmatrix} S & 0 \\ 0 & 0 \end{smallmatrix}\right]} \leq 0 \end{cases}$$

Therefore, the matrices $(\bar{P}, Q, R, T = \begin{bmatrix} S & 0 \\ 0 & 0 \end{bmatrix})$ must be the feasible solutions of (20).

Thirdly, if the matrices $(\bar{P}, Q, R, T = \begin{bmatrix} S & S_2 \\ S_3 & S_4 \end{bmatrix})$ are feasible solutions of (20), then

$$\begin{cases} \Phi_3 \geq 0 \Longrightarrow \begin{bmatrix} \Phi_1 & \begin{bmatrix} 0 & S_2 \\ S_3^T & 0 \end{bmatrix} \\ * & -\Pi_2 \end{bmatrix} \geq 0 \Longrightarrow \Phi_1 \geq 0 \\ \Phi_4 \leq 0 \Longrightarrow \bar{\Phi}_4 = \begin{bmatrix} \Phi_2 & \Pi_1 \\ * & \Pi_2 \end{bmatrix} \leq 0 \Longrightarrow \Phi_2 \leq 0 \end{cases}$$

Therefore, the matrices $(\bar{P}, Q, R, S)$ must be the feasible solutions of (19).

By combining the above three steps, the statement of Proposition 2 is proved. ■

Remark 5: Proposition 2 shows that the WBI does not improve the JBI-based BRL when the non-augmented Lyapunov functional (29) is used. In [8]–[10], the tighter advantage of the WBI compared with the JBI was successfully found when the augmented Lyapunov functional was used. Therefore, the tighter inequality does not always lead to a less conservative criterion, and the Lyapunov functional is another important factor linked to the conservatism.

The second finding is that *a tighter inequality may lead to a more conservative criterion, even under the same Lyapunov function*. It is concluded from the conservatism comparison of Theorem 1.C3 and Theorem 2.C1, which are all obtained by $V_a(k)$. Although the WBI is tighter than GFMBI as mentioned in Proposition 1, it can be proved that Theorem 2.C1 is less conservative than Theorem 1.C3, summarized as follows.

Proposition 3: For any given h and γ, if there exist feasible solutions of LMI (21), then there must exist feasible solutions of LMI (31); when there is no feasible solution of LMI (21) for given h and γ, there may still exist feasible solutions of LMI (31) for the same h and γ.

Proof: Firstly, $\Phi_{8,j}$ of Theorem 2.C1 is equivalent to (37), i.e.,

$$\Phi_{8,j} \leq 0 \Leftrightarrow \bar{\Phi}_{8,j} = \Phi_{5,j} - \mathfrak{E}_{4-j}^T (\tilde{R} - T_j \tilde{R}^{-1} T_j^T) \mathfrak{E}_{4-j} \leq 0$$

On the one hand, for any given h and γ, the feasible solutions, (P, Q, R, T), of (21) in Theorem 1.C3 lead to

$$\left.\begin{array}{r} \Phi_5 \leq 0 \Longrightarrow \Phi_{5,j} \leq 0 \\ \Phi_3 \geq 0 \Longrightarrow \tilde{R} - T_j \tilde{R}^{-1} T_j^T \geq 0 \end{array}\right\} \Longrightarrow \bar{\Phi}_{8,j} \leq 0 \Longrightarrow \Phi_{8,j} \leq 0$$

Thus, the matrices (P, Q, R, T) must be the feasible solutions of (31) for the same h and γ.

On the other hand, when there is no feasible solution of (21) for given h and γ, namely, for all possible combinations of matrices (P, Q, R, T), no one can lead that

$$\Phi_{5,j} \leq 0 \tag{39}$$

However, it can be predicted that there may still existing one or more sets of matrices, (P, Q, R, T), satisfying the following condition

$$\Phi_{5,j} \leq \mathfrak{E}_{4-j}^T (\tilde{R} - T_j \tilde{R}^{-1} T_j^T) \mathfrak{E}_{4-j} \tag{40}$$

which means $\bar{\Phi}_{8,j} \leq 0$, thus, $\Phi_{8,j} \leq 0$. Thus, the matrices that do not satisfy (39) but satisfy (40) are the feasible solutions of (31) in Theorem 2.C1. ■

Remark 6: Although WBI (8) is tighter than GFMBI (13), Proposition 3 shows that Theorem 2.C1 obtained based on GFMBI (13) is less conservative than Theorem 1.C3 derived by WBI (8). This finding is opposite to the existing experience that tighter inequalities lead to less conservative results [10].

Remark 7: An important issue arises from the aforementioned discussion, namely, it seems to be not enough to directly judge the conservatism of the resulting criteria only based on the tightness of the inequality applied. In fact, the research of [35] shows that equivalent criteria may be established from different Lyapunov functions and estimation methods.

IV. Numerical Example

A numerical example is applied to verify the main results of this note. The conservatism of the BRLs is checked via the calculated optimal H_∞ performance index (OHPI), and the criterion providing smaller OHPI is less conservative [27]. As mentioned in Remark 1, the aim of the note is not to derive a BRL with as small conservatism as possible. The system studied is very simple and has not been studied in the literature, thus no result reported in the literature is given. On the contrary, the results of Theorem 2 based on the proposed inequality are compared with that of Theorem 1 by the existing inequalities to show the advantages of the proposed inequality.

Example 1: Consider system (1) with the parameters

$$A = \begin{bmatrix} 0.8 & 0 \\ 0.05 & 0.9 \end{bmatrix}, A_d = \begin{bmatrix} -0.1 & 0 \\ -0.2 & -0.1 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, C = [1 \ \ 0]$$

The values of OHPIs with respect to different h calculated by the BRLs in Theorems 1 and 2 are listed in Table I, where Th. indicates Theorem, the SLF and the ALF indicate simple Lyapunov functional (29) and augmented Lyapunov functional (30), respectively, the JBI, the WBI, and the FMBI indicate inequalities (6), (8), and (11), respectively, and GFMBI1 and GFMBI2 indicate inequality (13) with different g_i (see the proof of Theorem 2 for details). Moreover, the number of decision variables (NoVs) is also listed in the table.

The results listed in the table commendably validate the statements of Remarks 3 and 4 and Propositions 2 and 3.

TABLE I
OHPIs FOR VARIOUS h (N/A INDICATES THE BRL IS UNAVAILABLE).

BRLs	h				NoVs
	5	10	15	20	
Th. 1.C1 (SLF+JBI)	4.157	6.488	13.333	N/A	13
Th. 1.C2 (SLF+WBI)	4.157	6.488	13.333	N/A	25
Th. 1.C3 (ALF+WBI)	4.142	6.437	9.664	17.855	32
Th. 1.C4 (ALF+FMBI)	4.053	6.507	9.330	13.651	220
Th. 2.C1 (ALF+GFMBI1)	4.077	6.418	9.213	14.732	32
Th. 2.C2 (ALF+GFMBI2)	4.031	6.403	9.091	13.251	96

- The advantages of the proposed inequality compared with the existing ones are shown. On one hand, Theorem 2.C1 provides smaller OHPIs with the same NoV in comparison with Theorem 1.C3, which verifies the statement of Remark 3; On the other hand, Theorem 2.C2 provides the least conservative results, which verifies the statement of Remark 4.
- The first finding summarized in Section III.C (above Proposition 2) is verified. On one hand, Theorem 1.C1 and C2 lead to the same OHPIs, which matches the statement of Proposition 2. On the other hand, Theorem 1.C3 provides smaller OHPIs than Theorem 1.C2, which shows that the tighter advantage of the WBI is revealed if the augmented Lyapunov functional is used. Those observations verify the statement of Remark 5;
- The second finding summarized in Section III.C (above Proposition 3) is verified. The OHPIs provided by Theorem 2.C1 are smaller than those of Theorem 1.C3, which matches the statement of Proposition 3 and also verifies the statement of Remark 6.

V. Conclusions

This note has discussed the relationship between the tightness of summation inequality and the conservatism of corresponding criterion based on the H_∞ performance analysis of a discrete-time system with a time-varying delay. Based on the theoretical analysis on both the tightness comparison and the conservatism comparison, two interesting phenomenons unlike the previous experience have been found. Firstly, compared with widely used JBI, the tighter WBI does not always leads to a less conservative criterion and the embodiment of tighter advantage is related to the Lyapunov functional. Secondly, the WBI is tighter than the GFMBI, while the criterion derived based on the GFMBI is less conservative. Finally, a numerical example has been given to verify those findings.

References

[1] H. Shao and Q.L. Han. “New stability criteria for linear discrete-time systems with interval-like time-varying delays,” *IEEE Trans. Autom. Control*, vol. 56, no. 3, pp. 619-625, 2011.

[2] J. Liu and J. Zhang J. “Note on stability of discrete-time time-varying delay systems,” *IET Control Theory Appl.*, vol. 6, no. 2, pp. 335-339, 2012.

[3] H. Gao and T. Chen. “New results on stability of discrete-time systems with time-varying state delay,” *IEEE Trans. Autom. Control*, vol. 52, no. 2, pp. 328-334, 2007.

[4] B. Zhang, S. Xu, and Y. Zou. “Improved stability criterion and its applications in delayed controller design for discrete-time systems,” *Automatica*, vol. 44, no. 11, pp. 2963-2967, 2008.

[5] Y. He, M. Wu, G.P. Liu, and J.H. She. “Output feedback stabilization for a discrete-time systems with a time-varying delay,” *IEEE Trans. Autom. Control*, vol. 53, no. 10, pp. 2372-2377, 2008.

[6] X. Meng, J. Lam, B. Du, and H. Gao. “A delay-partitioning approach to the stability analysis of discrete-time systems,” *Automatica*, vol. 46, no. 3, pp. 610-614, 2010.

[7] X. Jiang, Q.L. Han, and X. Yu. “Stability criteria for linear discrete-time systems with interval-like time-varying delay,” In *Proceedings of the 2005 American control conference*, Portland, OR, USA.

[8] P.T. Nam, P.N. Pathirana, and H. Trinh. “Discrete wirtinger-based inequality and its application,” *J. Frankln Inst.*, vol. 352, no. 5, pp. 1893-1905, 2015.

[9] A. Seuret, F. Gouaisbaut, and E. Fridman. “Stability of discrete-time systems with time-varying delays via a novel summation inequality,” *IEEE Trans. Autom. Control*, vol. 60, no. 10, pp. 2740-2745, 2015.

[10] X.M. Zhang, and Q.L. Han. “Abel lemma-based finite-sum inequality and its application to stability analysis for linear discrete time-delay systems,” *Automatica*, vol. 57, pp. 199-202, 2015.

[11] P.T. Nam, H. Trinh, P.N. Pathirana “Discrete inequalities based on multiple auxiliary functions and their applications to stability analysis of time-delay systems,” *J. Frankln Inst.*, vol. 352, no. 12, pp. 5810-5831, 2015

[12] J. Chen, J. Lu, S. Xu. “Summation inequality and its application to stability analysis for time-delay systems,” *IET Control Theory Appl.*, vol. 10, no. 4, pp. 391-395, Feb. 2016.

[13] C.K. Zhang, Y. He, L. Jiang, M. Wu, and H.B. Zeng, “Delay-variation-dependent stability of delayed discrete-time systems,” *IEEE Trans. Autom. Control*. DOI: 10.1109/TAC.2015.2503047.

[14] Q.M. Kwon, M.J. Park, J.H. Park, S.M. Lee, and E.J. Cha. “Improved robust stability criteria for uncertain discrete-time systems with interval time-varying delays via new zero equalities,” *IET Control Theory Appl.*, vol. 6, no. 16, pp. 2567-2575, 2012.

[15] Q.M. Kwon, M.J. Park, J.H. Park, S.M. Lee, and E.J. Cha. “Stability and stabilization for discrete-time systems with time-varying delays via augmented Lyapunov-Krasovskii functional,” *J. Frankln Inst.*, vol. 350, no. 3, pp. 521-540, 2013.

[16] C. Peng. “Improved delay-dependent stabilisation criteria for discrete systems with a new finite sum inequality,” *IET Control Theory Appl.*, vol. 6, no. 3, pp. 448-453, 2012.

[17] D. Yue, E. Tian, and Y. Zhang. “A piecewise analysis method to stability analysis of linear continuous/discrete systems with time-varying delay,” *Int. J. Robust Nonlinear Control*, vol. 19, no. 13, pp. 1493-1518, 2009.

[18] X. Xu, J. Lam, B. Zhang, and Y. Zou. “A new result on the delay-dependent stability of discrete systems with time-varying delays,” *Int. J. Robust Nonlinear Control*, vol. 24, no. 16, pp. 2512-2521, 2014.

[19] H. Huang and G. Feng. “Improved approach to delay-dependent stability analysis of discrete-time systems with time-varying delay,” *IET Control Theory Appl.*, vol. 4, no. 10, pp. 2152-2159, 2010.

[20] Z. Feng, J. Lam, and G.H. Yang. “Optimal partitioning method for stability analysis of continuous/discrete delay systems,” *Int. J. Robust Nonlinear Control*, vol. 25, no. 4, pp. 559-574, 2015.

[21] X.M. Sun, G.P. Liu, D. Rees, and W. Wang. “Delay-dependent stability for discrete-time systems with large delay sequence based on switching techniques,” *Automatica*, vol. 44, no. 11, pp. 2902-2908, 2008.

[22] Z.G. Wu, H.Y. Su, and J. Chu. “Robust stabilization for uncertain discrete singular systems with state delay,” *Int. J. Robust Nonlinear Contr.*, vol. 18, no. 16, pp. 1532-1550, 2008.

[23] S.H. Song, and J.K. Kim. “H_∞ control of discrete-time linear systems with norm-bounded uncertainties and time delay in state,” *Automatica*, vol. 34, no. 1, pp. 137-139, 1998.

[24] Y. He, M. Wu, Q.L. Han, and J. She. “Delay-dependent H_∞ control of linear discrete-time systems with an interval-like time-varying delay,” *Int. J. Syst. Sci.*, vol. 39, no. 4, pp. 427-436, 2008.

[25] X.M. Zhang, and Q.L. Han. “A new finite sum inequality approach to delay-dependent H_∞ control of discrete-time systems with time-varying delay,” *Int. J. Robust Nonlinear Contr.*, vol. 18, no. 6, pp. 630-647, 2008.

[26] S. Kim. “Relaxed inequality approach to robust H_∞ stability analysis of discrete-time systems with time-varying delay,” *IET Control Theory Appl.*, vol. 6, no. 13, pp. 2149-2156, 2012.

[27] S. Kim. “Further results on stability analysis of discrete-time systems with time-varying delays via the use of novel convex combination coefficients,” *Appl. Math. Comput.*, vol. 261, pp. 104-113, 2015.

[28] P. Park, J. Ko, and C. Jeong. “Reciprocally convex approach to stability of systems with time-varying delays,” *Automatica*, vol. 47, no. 1, pp. 235-238, 2011.

[29] P. Park, and J. Ko. “Stability and robust stability for systems with a time-varying delay,” *Automatica*, vol. 43, no. 10, pp. 1855-1858, 2007.

[30] A. Seuret and F. Gouaisbaut. “Wirtinger-based integral inequality: Application to time-delay systems,” *Automatica*, vol. 49, no. 9, pp. 2860-2866, 2013.

[31] H.B. Zeng, Y. He, M. Wu, and J. She. “Free-matrix-based integral inequality for stability analysis of systems with time-varying delay,” *IEEE Trans. Automat. Control*, vol. 60, no. 10, pp. 2768-2772, 2015.

[32] H.B. Zeng, Y. He, M. Wu, and J. She, “New results on stability analysis for systems with discrete distributed delay,” *Automatica*, vol. 60, pp. 189-192, 2015.

[33] C.K. Zhang, Y. He, L. Jiang, and M. Wu. “Stability analysis for delayed neural networks considering both conservativeness and complexity,” *IEEE Trans. Neural Netw. Learning Syst.*, vol. 27, no. 7, pp. 1486-1501, Jul. 2016.

[34] C.K. Zhang, Y. He, L. Jiang, M. Wu, and H.B. Zeng. “Stability analysis of systems with time-varying delay via relaxed integral inequalities,” *Systems & Control Letters*, vol. 92, pp. 52-61, 2016.

[35] S. Xu, and J. Lam, “On equivalence and efficiency of certain stability criteria for time-delay systems,” *IEEE Trans. on Automat. Control*, vol. 52, no. 1, pp. 95-101, 2007.